

Do Exercise 1 and four other problems. Cite any other students or resources which you used.

Exercise 1 (Short-answer essay questions). Computer-generated pictures of Julia sets reveal the rich geometry and dynamics of complex rational maps. They pose the challenge of explaining these experimental results in a satisfactory way. A picture may be well worth a thousand theorems.

These questions ask you to consider the qualitative significance of the results presented in class. There is no right answer for each prompt. Indeed, the purpose of this assignment is to consider various complementary explanations for a particular phenomenon.

For each question, explain the phenomenon in question by reference to precise mathematical statements.

- (a) Describe and explain some aspects about the appearance of Julia sets.
- (b) Explain how the dynamics of a rational map is “chaotic” on the Julia set.

Exercise 2. (a) Let X, Y be Riemann surfaces. Prove that a proper holomorphic map $X \rightarrow Y$ with no critical points is a covering map.

- (b) Let X be a Riemann surface. Prove that a family of holomorphic maps $\mathcal{F} = \{f : X \rightarrow \widehat{\mathbb{C}}\}$ is normal if and only if for each $K \subset X$ compact, $\max_{x \in K} \|f(x)\|$ is uniformly bounded over all $f \in \mathcal{F}$, where $\|\cdot\|$ is the spherical metric on $\widehat{\mathbb{C}}$.
- (c) Prove that a backwards-invariant closed set X with at least three points contains the Julia set.
- (d) Prove that if Ω is a connected component of the Fatou set of f , then $f(\Omega)$ a connected component of the Fatou set.
- (e) Consider the attracting basin $\mathcal{A}$ of an attracting periodic cycle. Using the Riemann-Hurwitz formula, prove that if $\mathcal{A}$ has only finitely many components, all of which are simply connected, then the period of the attracting cycle is at most 2. Prove that if all the components of $\mathcal{A}$ are infinitely connected, then the cycle is a fixed point.

Exercise 3 (Finite grand orbits). Let $f \in \mathbb{C}(z)$ be of degree $d \geq 2$.

- (a) Prove that f is a polynomial if and only if $f^{-1}(\infty) = \{\infty\}$.
- (b) Prove that $0, \infty$ are both exceptional points for f if and only if $f(z) = \alpha z^n$ for $\alpha \neq 0$ and $n = \pm d$.
- (c) Deduce that the exceptional set of f is nonempty if and only if f is conjugate under a Möbius transformation to either a polynomial or $z \mapsto 1/z^d$.

Exercise 4 (The main cardioid). This exercise deals with the largest bulb of the Mandelbrot set, the *main cardioid*.

- (a) Consider the family of quadratic maps $g_\lambda(z) = z^2 + \lambda z$ for $\lambda \in \mathbb{C}$. Find the two fixed points of f_λ in the complex plane and calculate their multipliers. Prove that g_λ has an attracting fixed point if and only if $\lambda \in \mathbb{D} \cup (2 + \mathbb{D})$.

- (b) (Optional) Make a picture of the set of λ for which the sequence $\{g_\lambda^n(0)\}_n$ stays bounded.
- (c) Prove that each g_λ is conjugate under an affine change of coordinates to a unique f_c , where $f_c(z) = z^2 + c$, and that the map $\lambda \mapsto c$ is a degree-2 branched cover.
Determine the set C of $c \in \mathbb{C}$ for which f_c has an attracting fixed point. Calculate the multiplier of the fixed point. Draw the boundary of C .
- (d) Prove that for all $c \in C$, the Julia set $J(f_c)$ is connected and the attracting basins $\mathcal{A}(\alpha)$ and $\mathcal{A}(\infty)$ of the unique attracting fixed point α and ∞ are both isomorphic to $\mathbb{D}$.
Draw some Julia sets of maps f_c for $c \in C$. What do you observe when c approaches the boundary ∂C ?

Exercise 5 (Real quadratic maps). This exercise deals with the intersection of the Mandelbrot set with the real axis.

- (a) Demonstrate that the set of $c \in \mathbb{R}$ for which $f_c(z) = z^2 + c$ has a connected Julia set is $[-2, 1/4]$.
- (b) Let $\alpha < \beta$ be the two fixed points of f_c . Prove that for $c \in [-2, 1/4]$, $K(f_c) \cap \mathbb{R} = [-\beta, \beta]$.
- (c) Show that the set of $c \in \mathbb{R}$ such that f_c has an attracting fixed point is $(-3/4, 1/4)$ and the set of $c \in \mathbb{R}$ such that f_c has a parabolic fixed point is $\{-3/4, 1/4\}$.
- (d) Show that the set of $c \in \mathbb{R}$ such that f_c has an attracting cycle of period two is $(-5/4, -3/4)$.

Exercise 6 (Rigidity of Julia sets).

- (a) For $|a| < 1$, let

$$\phi_a(z) = \frac{z - a}{1 - \bar{a}z}.$$

A *Blaschke product* of degree d is a rational map of the form

$$B(z) = e^{i\theta} \phi_{a_1}(z) \cdots \phi_{a_d}(z)$$

for all $|a_i| < 1$. Prove that a Blaschke product sends each of $\mathbb{D}$ and $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ to themselves.

Deduce that the Julia set of a Blaschke product is contained in the unit circle. Prove that if one of the a_i is equal to zero, the the Julia set is the entire unit circle.

- (b) The following is a theorem of Fatou.

Suppose that f is an analytic function on $\mathbb{D}$ and that $\lim_{|z| \rightarrow 1} |f(z)| = 1$. First prove that f can only have finitely many zeroes in $\mathbb{D}$. Using this fact, deduce that f must be a Blaschke product.

- (c) Prove that if f is a rational map for which the Julia set of f is the unit circle, then either f or $1/f$ is a Blaschke product.
- (d) The Chebyshev polynomials $P_n(z)$ are defined by the recursion $P_{n+1}(z) + P_{n-1}(z) = zP_n(z)$ with initial conditions $P_1(z) = z$ and $P_2(z) = z^2 - 2$. Let $\psi(z) = z + z^{-1}$. Prove that the following diagram commutes:

$$\begin{array}{ccc} \widehat{\mathbb{C}} & \xrightarrow{z \mapsto z^n} & \widehat{\mathbb{C}} \\ \psi \downarrow & & \downarrow \psi \\ \widehat{\mathbb{C}} & \xrightarrow{P_n} & \widehat{\mathbb{C}} \end{array}$$

Observe that ψ sends the unit circle to the interval $[-2, 2]$. Using these facts, prove that $J(P_n) = [-2, 2]$.

- (e) Suppose that f is a rational map of degree d for which $J(f) = [-2, 2]$. Prove that there exists a Blaschke product B of degree d for which the following diagram commutes:

$$\begin{array}{ccc} \widehat{\mathbb{C}} & \xrightarrow{B} & \widehat{\mathbb{C}} \\ \psi \downarrow & & \downarrow \psi \\ \widehat{\mathbb{C}} & \xrightarrow{f} & \widehat{\mathbb{C}} \end{array}$$

Show that if f is furthermore a polynomial of degree d , then $f = \pm P_d$.

Exercise 7 (Lattès maps). Let $\tau \in \mathbb{H}$ and $\Gamma = \mathbb{Z}[\tau]$.

- (i) Prove that for some $\lambda \in \mathbb{C}$, the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C}/\Gamma & \xrightarrow{z \mapsto z+1/2} & \mathbb{C}/\Gamma \\ \downarrow \wp_\tau & & \downarrow \wp_\tau \\ \widehat{\mathbb{C}} & \xrightarrow{z \mapsto \lambda/z} & \widehat{\mathbb{C}} \end{array}$$

- (ii) Recall that the Lattès map f_Γ was constructed according to the commutative diagram

$$\begin{array}{ccc} \mathbb{C}/\Gamma & \xrightarrow{z \mapsto 2z} & \mathbb{C}/\Gamma \\ \downarrow \wp_\tau & & \downarrow \wp_\tau \\ \widehat{\mathbb{C}} & \xrightarrow{f_\Gamma} & \widehat{\mathbb{C}} \end{array}$$

Prove that f_Γ has poles at $\infty, 0, 1, \lambda$ and double zeroes at $\pm\sqrt{\lambda}$. Show that the multiplier at ∞ is 4 and therefore that

$$f_\Gamma(z) = \frac{(z^2 - \lambda)^2}{4z(z-1)(z-\lambda)}.$$

- (iii) (Optional) Using the ideas from Exercise 5 of Homework 1, prove that the map $\tau \mapsto \lambda = \lambda(\tau)$ omits $\{0, 1\}$ and satisfies the equations

$$\lambda(\tau + 1) = 1/\lambda(\tau), \quad \lambda(-1/\tau) = 1 - \lambda(\tau).$$

Remark. The map $\lambda : \mathbb{H} \rightarrow \mathbb{C} \setminus \{0, 1\}$ is the universal covering of the thrice-punctured sphere.

Exercise 8 (Shapes of Julia sets). This exercise presents a result of D. Calegari.

- (a) Let (X, d) be a metric space. We will write $d(x, A)$ for $\inf_{a \in A} d(x, a)$. For any two nonempty subsets A, B of X , define

$$d_H(A, B) = \max \left(\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right).$$

Prove that d_H makes the set of nonempty compact subset of X into a metric space. It is called the *Hausdorff metric*. Prove that any compact subset can be approximated by finite sets.

- (b) Let $K \subset \widehat{\mathbb{C}}$ be any nonempty and compact proper subset. We can suppose without loss of generality that $\infty \notin K$. For $p \notin K$, show that $f(z) = C(z-p)^N + p$ is disjoint from F and F lies in the attracting basin of ∞ for f .
- (c) Choose a finite set of points Y which approximate K in the Hausdorff metric. Prove that

$$g_\delta(z) := f(z) \prod_{y \in Y} \frac{z - (y + \delta)}{z - (y - \delta)}$$

converges uniformly on $\widehat{\mathbb{C}} \setminus Y$ to f . Show that as $\delta \rightarrow 0$, the Julia sets $J(g_\delta)$ converge in the Hausdorff topology to $J(f) \cup \bigcup_{n=0}^{\infty} f^{-1}(Y)$.

- (d) Prove that K can be approximated arbitrarily well in the Hausdorff topology by Julia sets of rational maps.

Exercise 9 (Asymptotic self-similar limits).

- (a) Do part (a) of Exercise 8.
- (b) Suppose that $f(z) \in \mathbb{C}(z)$ has a repelling fixed point at zero of multiplier λ . Prove that there exists a closed set $K \subset \mathbb{C}$ which is fully invariant under multiplication by λ , i.e. $\lambda \cdot K = K$, and such that for any compact set $E \subset \mathbb{C}$, the sets $(\lambda^n \cdot J(f)) \cap E$ converge in the Hausdorff topology to $K \cap E$. (Hint: Consider the global Koenigs map.)

The set K is the *asymptotic self-similar limit* of $J(f)$ at the repelling fixed point.

- (c) Consider an open set U around zero so that $f^{-1}(U) \subset U$. Prove that the quotient $(U \setminus \{0\}) / \sim$ under the relation $f(z) \sim z$ is isomorphic to the torus $\mathbb{C}^\times / \langle \lambda \rangle$, and that $K \setminus \{0\}$ projects down to a subset of this torus.

Exercise 10. For this exercise you will draw Julia sets without using pre-existing fractal drawing software.

Pick three interesting rational maps. For each of them, draw the Julia set by plotting iterated preimages of a chosen point in the Julia set. Why is this a better strategy than trying to pick a point on it and plotting iterated forward images? Compare your picture to one made by a fractal software.

Exercise 11. Pick any exercise which you did not already complete from a previous problem set.